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Variational formulations for steady water waves with vorticity

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For free-surface water flows with a vorticity that is monotone with depth, we show that any critical point of a functional representing the total energy of the flow adjusted with a measure of the vorticity, subject to the constraints of fixed mass and horizontal momentum, is a steady water wave.

1. Introduction

We provide two new variational formulations of steady periodic (inviscid, incompressible) water waves with vorticity propagating over a flat bed. Both formulations are based on standard elementary principles, but the Lagrangians and the way they are treated are new. Both Lagrangians lead precisely to the governing equations for steady water waves. Our first, more fundamental, variational principle is expressed entirely in terms of the natural invariants (energy, mass, momentum and vorticity) in the physical coordinates and requires the vorticity to vary monotonically with the depth. It cannot be specialized to the irrotational case. The second variational principle does specialize to the irrotational case, but is different from other known principles because its formulation depends on the choice of coordinates using the streamfunction.

In the irrotational case, variational formulations for steady water waves have a very long history. First, the equations of motion with the appropriate boundary conditions for irrotational water waves were derived by Luke (1967) from a variational principle, whose Lagrangian is based on the velocity potential, which of course does not exist in the presence of vorticity. Shortly thereafter, a Hamiltonian formulation of the irrotational problem was introduced by Zakharov (1968) and later pursued by Miles (1977) and Milder (1977). A different case had been treated much earlier by Friedrichs (1933).

In this paper, we are assuming there is no surface tension. In the presence of surface tension, there are other variational principles for steady water waves that employ the spatial dynamics method, which treats the horizontal spatial variable as the dynamic variable (see for example Mielke 1991; Bridges 1992; Groves & Toland 1997). The existence proofs that use these principles require non-vanishing surface tension.

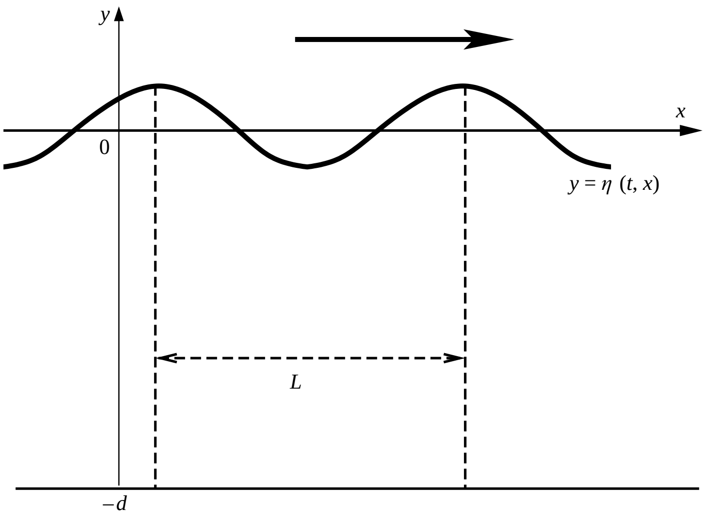

FIGURE 1. The profile of a steady periodic wave.

There is also an immense body of literature on variational formulations of the various classical small-amplitude long-wave approximations to the governing equations – the shallow-water, Boussinesq and Korteweg–de Vries equations all emerge from this process (see for example Benjamin 1984; Craig & Groves 1994).

Variational principles can be used for various purposes: to construct solutions analytically, for numerical computations, or to analyse the stability of the waves. Large classes of exact water waves with vorticity have been constructed by analytical methods in Constantin & Strauss (2004); and a project is in preparation to construct them numerically using the second variational principle presented in this paper. As for the question of stability, there is the classical Benjamin–Feir instability (Benjamin & Feir 1967; Bridges & Mielke). In work in preparation (Constantin & Strauss 2005), it will be shown how both variational formulations presented in this paper shed light on the stability problem.

In the next two sections we introduce the two formulations. We also indicate how they are formally dual to each other provided that the vorticity varies monotonically with the depth and the free surface is fixed. In the Appendix we provide some standard background material.

2. Basic equations

In order to fix notation, we list here the basic equations of motion of two-dimensional incompressible inviscid flow, possibly rotational. See the Appendix for details. Let x and y be the horizontal and vertical coordinates, respectively. Let $y = -d$ be the flat bed and $y = \eta(t, x)$ the free surface. Let ψ be the streamfunction and ω the scalar vorticity. We assume periodicity in the variable x with a given period (figure 1).

It is well-known (Longuet-Higgins 1983) that the following functionals are invariants of the flow: the total energy

$$\mathscr{E} = \int\int \left[\frac{u^2 + v^2}{2} + g(y + d)\right] \mathrm{d}y \, \mathrm{d}x, \qquad (1)$$

the fluid mass

$$m = \int\int \mathrm{d}y\,\mathrm{d}x, \tag{2}$$

the horizontal component of the momentum,

$$\mathscr{M} = \int\int u\,\mathrm{d}y\,\mathrm{d}x, \tag{3}$$

and the integral

$$\mathscr{F} = \int\int F(\omega)\,\mathrm{d}y\,\mathrm{d}x, \tag{4}$$

for an arbitrary function F. All of the integrals above are taken over a periodic section of the fluid domain. (There are other invariants, but we do not have occasion to use them.)

If we consider a steady wave travelling at a given speed $c > 0$, then ω and $\psi - cy$ are functionally dependent and we write

$$\omega = \gamma(\psi - cy). \tag{5}$$

The vorticity function γ could be multi-valued but we assume it to be single-valued. Naturally, we use a moving frame with $x - ct$ replaced by x. In terms of ψ, the governing equations are (see the Appendix)

$$\left.\begin{array}{l} \Delta\psi = -\gamma(\psi - cy) \qquad \text{in} \quad -d < y < \eta(x), \\ |\nabla(\psi - cy)|^2 + 2g(y+d) = Q \qquad \text{on} \quad y = \eta(x), \\ \psi - cy = k_S \qquad \text{on} \quad y = \eta(x), \\ \psi - cy = k_B \qquad \text{on} \quad y = -d, \end{array}\right\} \tag{6}$$

where k_S and k_B are constants. The difference of these two constants is

$$k_S - k_B = \int_{-d}^{\eta(x)} (u - c)\,\mathrm{d}y = p_0,$$

which is called the relative mass flux and is independent of x. The nonlinear boundary condition at $y = \eta(x)$ is an expression of Bernoulli's law (see Constantin & Strauss 2004). The constant Q can be expressed as $Q = 2(E + gd)$, where E is the hydraulic head of the flow.

3. First variational formulation

Consider periodic steady two-dimensional waves propagating at the free surface of a water layer over a flat bed $y = -d$. Let $F : \mathbb{R} \to \mathbb{R}$ be a C^2-function for which F'' vanishes nowhere. That is, F is either strictly convex or strictly concave. We define the C^1 function γ by

$$\gamma = (F')^{-1}. \tag{7}$$

(Alternatively, we could begin with any given C^1 function γ with $\gamma \neq 0$ and then define F up to a constant by (7).) Now the streamfunction ψ, given up to a constant by (6), and the free-surface profile η completely determine the steady flow.

The functionals $\mathscr{E}$, $\mathscr{F}$, m and $\mathscr{M}$ may be written entirely in terms of ψ and η. They are defined on the linear space of functions

$$(\psi, \eta) \in C^2_{per}(\mathbb{R} \times [-d, \infty)) \times C^1_{per}(\mathbb{R})$$

that are smooth of period L in the variable x. Note that ψ is defined for $-d \leqslant y < \infty$, part of which lies above the fluid domain. Normally, ψ and η are coupled via the boundary conditions. However, if the existence of a critical point is not an issue, and if the free boundary is regular, then we can extend ψ across the free boundary and we are allowed to consider independent variations with respect to these two functions. The main aim of this paper is to prove the following variational characterization of steady water waves with vorticity.

THEOREM 1. *Any critical point of $\mathscr{E} - \mathscr{F}$, subject to the constraints of fixed mass m and horizontal momentum $\mathscr{M}$, is a steady periodic water wave with vorticity function γ.*

Remark. We could alternatively take $\mathscr{F}$ as a third constraint and look for critical points of the energy $\mathscr{E}$. Then there would be an additional Lagrange multiplier for $\mathscr{F}$.

Proof. By definition, a critical point satisfies the Euler–Lagrange equation

$$\delta(\mathscr{E} - \mathscr{F}) = \lambda\, \delta\mathscr{M} + \mu\, \delta m, \tag{8}$$

where λ and μ are Lagrange multipiers. Let (ψ_1, η_1) denote a perturbation of (ψ, η) in the specified function space. Let $\omega_1 = -\Delta\psi_1$. Denote by $\Omega = \{0 < x < L, -d < y < \eta(x)\}$ one period of the fluid domain, by S its top boundary $\{y = \eta(x)\}$, and by B its bottom boundary $\{y = -d\}$. We have

$$\delta m(\psi, \eta)(\psi_1, \eta_1) = \lim_{\varepsilon\to 0} \frac{m(\psi + \varepsilon\psi_1, \eta + \varepsilon\eta_1) - m(\psi, \eta)}{\varepsilon} = \int_S \eta_1\, \mathrm{d}x,$$

while

$$\delta\mathscr{M}(\psi, \eta)(\psi_1, \eta_1) = \int\int_\Omega u_1\, \mathrm{d}y\, \mathrm{d}x + \int_S u\, \eta_1\, \mathrm{d}x,$$

$$\delta\mathscr{F}(\psi, \eta)(\psi_1, \eta_1) = \int\int_\Omega F'(\omega)\, \omega_1\, \mathrm{d}y\, \mathrm{d}x + \int_S F(\omega)\, \eta_1\, \mathrm{d}x,$$

$$\delta\mathscr{E}(\psi, \eta)(\psi_1, \eta_1) = \int_S \left(\frac{|\nabla\psi|^2}{2} + g(\eta + d)\right) \eta_1\, \mathrm{d}x + \int\int_\Omega \nabla\psi \cdot \nabla\psi_1\, \mathrm{d}y\, \mathrm{d}x$$

$$= \int_S \left\{\left[\frac{|\nabla\psi|^2}{2} + g(\eta + d)\right] \eta_1 + \psi\, (\psi_{1y} - \eta_x\, \psi_{1x})\right\} \mathrm{d}x - \int\int_\Omega \psi\, \Delta\psi_1\, \mathrm{d}y\, \mathrm{d}x - \int_B \psi\psi_{1y}\, \mathrm{d}x,$$

by Green's theorem. By definition, any critical point (ψ, η) satisfies (8), so that

$$\int_S \left\{\frac{|\nabla\psi|^2}{2} + g(\eta + d) - \lambda u - \mu - F(\omega)\right\} \eta_1\, \mathrm{d}x + \int_S \psi\, (\psi_{1y} - \eta_x\, \psi_{1x})\, \mathrm{d}x$$

$$- \int_B \psi\psi_{1y}\, \mathrm{d}x + \int\int_\Omega \{-\psi\, \Delta\psi_1 - \lambda\psi_{1y} - F'(\omega)\, \omega_1\}\, \mathrm{d}y\, \mathrm{d}x = 0.$$

It is convenient to rewrite the preceding identity as

$$\int_S \left\{\frac{|\nabla(\psi - \lambda y)|^2}{2} + g(\eta + d) - \tfrac{1}{2}\lambda^2 - \mu - F(\omega)\right\} \eta_1\, \mathrm{d}x$$

$$+ \int_S (\psi - \lambda y)\, \frac{\partial\psi_1}{\partial n}\, \mathrm{d}l + \lambda \int_S y\, \frac{\partial\psi_1}{\partial n}\, \mathrm{d}l - \int_B \psi\psi_{1y}\, \mathrm{d}x$$

$$- \int\int_\Omega \{(\psi - \lambda y)\, \Delta\psi_1 + \lambda\, y\, \Delta\psi_1 + \lambda\, \psi_{1y} + F'(\omega)\, \omega_1\}\, \mathrm{d}y\, \mathrm{d}x = 0,$$

with n being the unit outer normal on the surface S and $\mathrm{d}l$ being the measure of arclength. Since four terms cancel in view of the identity $y\Delta\psi_1 + \psi_{1y} = \nabla\cdot(y\nabla\psi_1)$, or

$$\int\int_{\Omega}\{y\,\Delta\psi_1 + \psi_{1y}\}\,\mathrm{d}y\,\mathrm{d}x - \int_S y\,\frac{\partial\psi_1}{\partial n}\,\mathrm{d}l + \int_B y\psi_{1y}\,\mathrm{d}x = 0,$$

we obtain

$$\int_S \left\{\frac{|\nabla(\psi-\lambda\,y)|^2}{2} + g(\eta+d) - \tfrac{1}{2}\lambda^2 - \mu - F(\omega)\right\}\eta_1\,\mathrm{d}x - \int_B (\psi-\lambda y)\psi_{1y}\,\mathrm{d}x$$
$$+\int_S(\psi-\lambda\,y)\,\frac{\partial\psi_1}{\partial n}\,\mathrm{d}l - \int\int_{\Omega}\{(\psi-\lambda\,y)\,\Delta\psi_1 + F'(\omega)\omega_1\}\,\mathrm{d}y\,\mathrm{d}x = 0. \qquad (9)$$

We make four choices of the perturbation function. (i) We take $\eta_1 = 0$ and ψ_1 a solution of the elliptic problem

$$\Delta\psi_1 = -\omega_1 \quad \text{in} \quad \Omega,$$
$$\frac{\partial\psi_1}{\partial y} = 0 \quad \text{on} \quad B,$$
$$\frac{\partial\psi_1}{\partial n} = 0 \quad \text{on} \quad S,$$

so that (9) reduces to

$$\int\int_{\Omega}\{\psi - \lambda\,y - F'(\omega)\}\,\omega_1\,\mathrm{d}y\,\mathrm{d}x = 0.$$

This is valid for all smooth functions ω_1 with $\int\int_{\Omega}\omega_1\,\mathrm{d}x\,\mathrm{d}y = 0$. Therefore,

$$\psi - \lambda\,y = F'(\omega) + k \qquad \text{in} \quad \Omega, \qquad (10)$$

for some constant k. This by (7) yields $\omega = \gamma(\psi - \lambda\,y - k)$. We may redefine ψ so that $k = 0$. Thus,

$$\Delta\psi = -\gamma(\psi - \lambda\,y) \qquad \text{in} \quad \Omega, \qquad (11)$$

since $\Delta\psi = -\omega$. This is the first equation in (6) with $c = \lambda$. Note that in the irrotational case, we would merely obtain $\psi - cy = \text{constant}$, which is a trivial flow.

(ii) Taking $\eta_1 = 0$ and defining ψ_1 as the solution of the elliptic problem

$$\Delta\psi_1 = 0 \quad \text{in} \quad \Omega,$$
$$\frac{\partial\psi_1}{\partial y} = 0 \quad \text{on} \quad B,$$
$$\frac{\partial\psi_1}{\partial n} = f \quad \text{on} \quad S,$$

where f is an arbitrary smooth function defined on S with $\int_S f\,\mathrm{d}l = 0$, identity (9) reduces to

$$\int_S(\psi - \lambda\,y)\,f\,\mathrm{d}l = 0.$$

Thus,

$$\psi - \lambda\,y = k_S \quad \text{on} \quad S, \qquad (12)$$

where k_S is a constant.

(iii) Taking $\eta_1 = 0$ and defining ψ_1 as the solution of the elliptic problem

$$\begin{aligned} \Delta\psi_1 &= 0 \qquad \text{in} \quad \Omega, \\ \frac{\partial \psi_1}{\partial y} &= f \qquad \text{on} \quad B, \\ \frac{\partial \psi_1}{\partial n} &= 0 \qquad \text{on} \quad S, \end{aligned}$$

where f is an arbitrary smooth function defined on B with $\int_B f \,\mathrm{d}x = 0$, identity (9) reduces to

$$\int_B (\psi - \lambda\, y)\, f \,\mathrm{d}x = 0.$$

Thus,

$$\psi - \lambda\, y = k_B \quad \text{on} \quad B,$$

where k_B is another constant.

(iv) Choosing $\psi_1 = 0$ throughout Ω in (9), with η_1 arbitrary, we obtain

$$\int_S \left\{ \frac{|\nabla(\psi - \lambda\, y)|^2}{2} + g(\eta + d) - \tfrac{1}{2}\lambda^2 - \mu - F(\omega) \right\} \eta_1 \,\mathrm{d}x = 0$$

for all smooth functions η_1. Hence,

$$\frac{|\nabla(\psi - \lambda\, y)|^2}{2} + g(\eta + d) - \tfrac{1}{2}\lambda^2 - \mu - F(\omega) = 0 \qquad \text{on} \quad S. \tag{13}$$

Notice that $F'(\omega) = k_S$ on S by (10) and (12). Since F' is strictly monotone, the only possibility is that ω is constant on S, say, $\omega = \omega_0$ on S. Therefore, (13) becomes

$$\frac{|\nabla(\psi - \lambda\, y)|^2}{2} + g(\eta + d) = \tfrac{1}{2} Q \qquad \text{on} \quad y = \eta(x), \tag{14}$$

with $Q = \lambda^2 + 2\mu + 2\,F(\omega_0)$. This is the second equation in (6). We have thus recovered precisely the governing equations (6) for steady water waves with vorticity ω and wave speed $\lambda = c$. □

Remark. We can summarize theorem 1 by stating that the equations of motion for steady water waves are obtained by taking variations of the functional

$$\mathscr{H}(\psi,\, \eta) = \int\!\!\int_\Omega \left\{ \frac{|\nabla(\psi - cy)|^2}{2} + g(y + d) - \tfrac{1}{2} Q - F(-\Delta\psi) \right\} \mathrm{d}y \,\mathrm{d}x. \tag{15}$$

Remark. For steady flows for which $u \neq c$, meaning that there are no stagnation points, the monotonicity of γ is equivalent to the monotonicity of the vorticity as a function of the depth. This follows immediately by differentiating (5) to obtain the identity $\omega_y = (u - c)\; \gamma'(\psi - cy)$. (see figure 2.)

4. Second variational formulation

Given any C^1 function γ, we introduce another functional $\mathscr{L}$ by

$$\mathscr{L}(\psi) = \int\!\!\int_\Omega \left\{ \frac{|\nabla(\psi - c^2 y)|^2}{2} - g(y + d) + \tfrac{1}{2} Q + \Gamma(cy - \psi) \right\} \mathrm{d}y \,\mathrm{d}x, \tag{16}$$

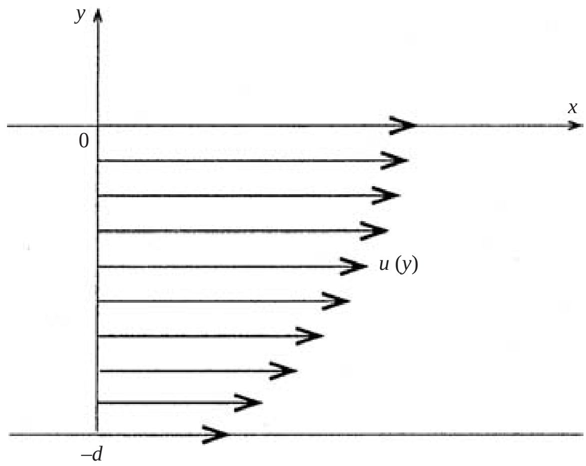

FIGURE 2. The vorticity of the flow with velocity field $(-2y^2 + 3d^2, 0)$ is zero at the flat surface and decreases strictly with depth.

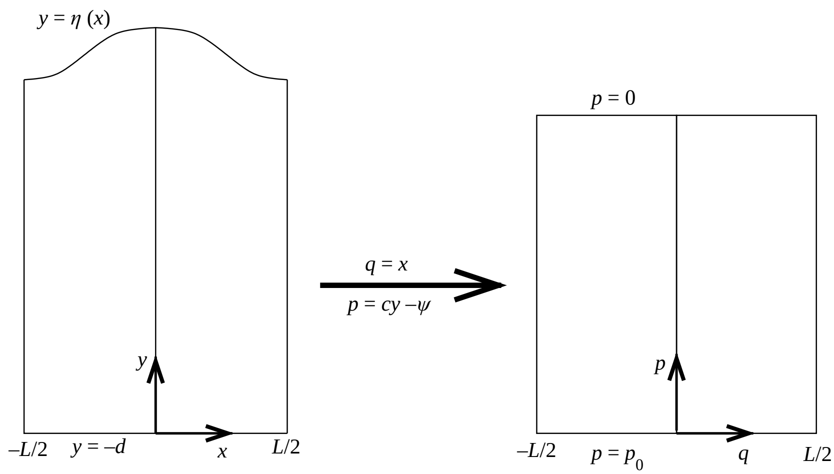

FIGURE 3. A coordinate transformation.

where

$$\Gamma(p) = \int_0^p \gamma(-s)\,\mathrm{d}s, \qquad p \in [p_0, 0]. \tag{17}$$

At the end of this section we will show that (16) and (17) are in duality.

Assuming that $\psi_y < c$ in $\overline{\Omega}$, let us write $\mathscr{L}$ as a functional of the height $h = y + d$ above the flat bed, expressed in terms of the new space variables

$$q = x, \qquad p = cy - \psi \tag{18}$$

(see figure 3). Notice that $F''(\omega)\,\omega_y = \psi_y - c < 0$ so that (18) is a local change of variables. It is actually a global change of variables (see Constantin & Strauss 2004).

Following Dubreil-Jacotin (1934), we can also transform the steady water-wave problem (6) to these variables to obtain

$$\left.\begin{aligned}&\left(1+h_q^2\right)h_{pp}-2h_ph_qh_{pq}+h_p^2h_{qq}+\gamma(-p)\,h_p^3=0 \quad \text{in} \quad p_0<p<0,\\ &1+h_q^2+(2gh-Q)\,h_p^2=0 \quad \text{on} \quad p=0,\\ &h=0 \quad \text{on} \quad p=p_0,\end{aligned}\right\} \tag{19}$$

with h even and of period L in the q variable. This is an elliptic equation (since $h_p>0$) with a nonlinear boundary condition. The fluid domain Ω, which depends on η, is transformed into the fixed rectangle $R=[0,L]\times[p_0,0]$. Since

$$h_q=\frac{\psi_x}{c-\psi_y}, \quad h_p=\frac{1}{c-\psi_y},$$

the functional $\mathscr{L}$ takes the form

$$\mathscr{J}(h)=\int\int_R\left\{\frac{1+h_q^2}{2h_p^2}-gh+\tfrac{1}{2}Q+\Gamma(p)\right\}h_p\,\mathrm{d}q\,\mathrm{d}p, \tag{20}$$

in terms of the (q,p)-variables. The domain of $\mathscr{J}$ is the set of all $h\in C^2(\overline{R})$ that are L-periodic in q with $h=0$ for $p=p_0$ and with $h_p\neq 0$ in $\overline{R}$. In this setting we obtain the second variational characterization of steady water waves with an arbitrary vorticity function.

THEOREM 2. *Fix the rectangle R, the parameter Q, and any C^1 function γ. Then any critical point of $\mathscr{J}$ is a steady periodic water wave with vorticity function γ.*

Proof. Given a perturbation k with $k=0$ for $p=p_0$, we have

$$\begin{aligned}\mathscr{J}(h+\varepsilon k)=\int\int_R\bigg\{&\frac{1+(h_q+\varepsilon k_q)^2}{2(h_p+\varepsilon k_p)}-g(h+\varepsilon k)(h_p+\varepsilon k_p)\\ &+\tfrac{1}{2}Q(h_p+\varepsilon k_p)+\Gamma(p)(h_p+\varepsilon k_p)\bigg\}\,\mathrm{d}q\,\mathrm{d}p.\end{aligned}$$

We obtain

$$\begin{aligned}\delta\mathscr{J}(h)(k)=\int\int_R\bigg\{&\frac{h_qk_q}{h_p}+\frac{1+h_q^2}{2h_p}\left(-\frac{k_p}{h_p}\right)-g\,hk_p-g\,h_pk\\ &+\tfrac{1}{2}Qk_p+\Gamma(p)\,k_p\bigg\}\,\mathrm{d}q\,\mathrm{d}p.\end{aligned}$$

By periodicity, integration by parts with respect to the q-variable does not produce boundary terms. On the other hand, the vanishing of k on $p=p_0$ leaves us with only boundary terms on $p=0$ after integration by parts with respect to the p-variable. Therefore,

$$\begin{aligned}\delta\mathscr{J}(h)(k)=&\int\int_R\left\{-\frac{h_{qq}}{h_p}+\frac{2h_qh_{pq}}{h_p^2}-\frac{\left(1+h_q^2\right)h_{pp}}{h_p^3}-\gamma(-p)\right\}k\,\mathrm{d}q\,\mathrm{d}p\\ &+\int_0^L\left\{-\frac{1+h_q^2}{2h_p^2}-gh+\tfrac{1}{2}Q\right\}k\,\mathrm{d}q\bigg|_{p=0},\end{aligned}$$

since $\Gamma(0)=0$. If h is a critical point, the above expression vanishes for all perturbations $k\in C^2(\overline{R})$ which are L-periodic in the q-variable and satisfy $k=0$ for $p=p_0$. The system (19) follows at once. □

Duality. The functionals $\mathscr{L}$ and $\mathscr{H}$ are not entirely unrelated. In fact, we conclude our discussion by showing that under the assumption $\gamma' < 0$, the functionals $\mathscr{L}$ and $\mathscr{H}$ are formally related via duality. Indeed, fix a domain $\Omega = \{(x, y) \in \mathbb{R}^2 : 0 < x < L,\ -d < y < \eta(x)\}$ with $d > 0$ and $\eta : \mathbb{R} \to \mathbb{R}$ of class C^1 and of period L. Let X be the Hilbert space of all functions obtained as the closure of the smooth functions $\Phi : \mathbb{R}^2 \to \mathbb{R}$ that are L-periodic in the q-variable and supported in the strip $\{(x, y) \in \mathbb{R}^2 :\ -d \leqslant y \leqslant \eta(x)\}$, with respect to the inner product

$$\langle \Phi_1, \Phi_2 \rangle = \int\int_\Omega \nabla\Phi_1 \cdot \nabla\Phi_2 \, \mathrm{d}y \, \mathrm{d}x + \int_B \Phi_1 \Phi_2 \, \mathrm{d}x.$$

Define the functional $L : X \to \mathbb{R}$ by

$$L(\Phi) = \int\int_\Omega \left\{ \tfrac{1}{2}|\nabla\Phi|^2 - g(y+d) + \tfrac{1}{2}Q + \Gamma(-\Phi) \right\} \mathrm{d}y \, \mathrm{d}x. \tag{21}$$

Denoting

$$\psi = \Phi + cy, \tag{22}$$

we see that

$$L(\Phi) = \mathscr{L}(\psi), \tag{23}$$

where $\mathscr{L}$ is given by (16). The dual functional $L^* : X \to \mathbb{R}$ is defined by

$$L^*(\Phi^*) = \sup_{\Phi \in X} \{ \langle \Phi^*, \Phi \rangle - L(\Phi) \}, \qquad \Phi^* \in X. \tag{24}$$

Since γ is given and bounded, Γ is uniformly Lipschitz so that $L : X \to \mathbb{R}$ is continuous. Since $\gamma' < 0$, we have $\Gamma'' > 0$ so that L is a strictly convex functional. Thus, for every fixed $\Phi^* \in X$, there is a unique $\Phi^0 \in X$ that attains the maximum, that is,

$$L^*(\Phi^*) = \langle \Phi^*, \Phi^0 \rangle - L(\Phi^0). \tag{25}$$

The function Φ^0 solves the Euler–Lagrange equation associated with the variational problem (24). In other words, in view of (21),

$$\int\int_\Omega \{\nabla\Phi^* \cdot \nabla\Phi - \nabla\Phi^0 \cdot \nabla\Phi + \gamma(\Phi^0)\Phi\} \, \mathrm{d}y \, \mathrm{d}x + \int_B \Phi^* \Phi \, \mathrm{d}x = 0, \tag{26}$$

for all $\Phi \in X$. On the other hand, $\partial_p [F(\gamma(-p))] = p\,\gamma'(-p)$ by (7). Since F is defined only up to a constant, we may specify $F(\gamma(0)) = 0$ and therefore

$$F(\gamma(-p)) = \int_0^p s\gamma'(-s) \, \mathrm{d}s = \Gamma(p) - p\,\gamma(-p). \tag{27}$$

By (25),

$$\begin{aligned} L^*(\Phi^*) = &\int_B \Phi^* \Phi^0 \, \mathrm{d}x \\ &+ \int\int_\Omega \left\{ \nabla\Phi^* \cdot \nabla\Phi^0 - \tfrac{1}{2}|\nabla\Phi^0|^2 + g(y+d) - \tfrac{1}{2}Q - \Gamma(-\Phi^0) \right\} \mathrm{d}y \, \mathrm{d}x. \end{aligned}$$

Thus, specializing (26) to $\Phi = \Phi^0$ and using (27), we obtain

$$\begin{aligned} L^*(\Phi^*) &= \int\int_\Omega \left\{ \tfrac{1}{2}|\nabla\Phi^0|^2 + g(y+d) - \tfrac{1}{2}Q - \Gamma(-\Phi^0) - \Phi^0\gamma(\Phi^0) \right\} \mathrm{d}y \, \mathrm{d}x \\ &= \int\int_\Omega \left\{ \tfrac{1}{2}|\nabla\Phi^0|^2 + g(y+d) - \tfrac{1}{2}Q - F(\gamma(\Phi^0)) \right\} \mathrm{d}y \, \mathrm{d}x. \end{aligned}$$

This expression is precisely $\mathscr{H}(\psi^0, \eta)$, if, in accordance with (15) and (22), we define $\psi^0 = \Phi^0 + cy$ and we let $\gamma(\Phi^0) = -\Delta\Phi^0$. This demonstrates that, when restricted to the set of critical points of $\mathscr{H}$ for a fixed domain, the functionals $\mathscr{H}$ and $\mathscr{L}$ are dual to each other. □

We emphasize that in theorem 2, although the vorticity function is arbitrary, the free surface is fixed while perturbations are restricted to the fluid velocity.

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Appendix. Equations of motion

The water waves that we typically see propagating on the surface of the sea (or on a river or lake) are, as a matter of common experience, locally approximately periodic and two-dimensional. That is, the motion is identical in any direction parallel to the crest line. To describe these waves, it suffices to consider a cross-section of the flow that is perpendicular to the crest line. Choose Cartesian coordinates (x, y) with the y-axis pointing vertically upwards and the x-axis being the direction of wave propagation, while the origin lies at the mean water level. Let $(u(t, x, y),\ v(t, x, y))$ be the velocity field of the flow, let $y = -d$ for some $d > 0$ be the flat bed, and let $y = \eta(t, x)$ be the water's free surface. Homogeneity (constant density) is a physically reasonable assumption for gravity waves (Lighthill 1978), and it implies the equation of mass conservation

$$u_x + v_y = 0. \tag{A 1}$$

Also appropriate for gravity waves is the assumption of inviscid flow (Lighthill 1978), so that the equation of motion is Euler's equation

$$\left.\begin{aligned} u_t + uu_x + vu_y &= -P_x, \\ v_t + uv_x + vv_y &= -P_y - g, \end{aligned}\right\} \tag{A 2}$$

where $P(t, x, y)$ denotes the pressure and g is the gravitational constant of acceleration. The free surface decouples the motion of the water from that of the air so that (Johnson 1997) the dynamic boundary condition

$$P = P_0 \qquad \text{on} \quad y = \eta(t, x), \tag{A 3}$$

must hold where P_0 is the atmospheric pressure. Moreover, since the same particles always form the free surface, we have the kinematic boundary condition

$$v = \eta_t + u\eta_x \qquad \text{on} \quad y = \eta(t, x). \tag{A 4}$$

On the flat bed we have the kinematic boundary condition

$$v = 0 \qquad \text{on} \quad y = -d, \tag{A 5}$$

expressing the fact that water cannot penetrate the rigid bed at $y = -d$. The governing equations for water waves (A 1)–(A 5) are of course highly nonlinear. We assume the flow is periodic, in the sense that the velocity field (u, v), the pressure P and the free surface η all have period (wavelength) L in the x-variable. Let

$$\Omega(t) = \{(x, y) \in \mathbb{R}^2 \ :\ 0 < x < L,\ -d < y < \eta(t, x)\},$$

be one period of the fluid domain at time $t \geqslant 0$. The restriction to $x \in (0, L)$ is justified by the periodicity assumption in the x-variable. For the solutions of (A 1)–(A 5) under consideration, we assume that $u,\ v,\ P,\ \eta$ are all C^2 in the variables (t, x, y) as $(x, y) \in \overline{\Omega(t)}$, where t varies in some time interval.

The incompressibility (A 1) enables us to introduce a streamfunction $\psi(t, x, y)$ defined up to a constant by

$$\psi_x = -v, \qquad \psi_y = u. \tag{A 6}$$

Thus,

$$\Delta\psi = -\omega, \tag{A 7}$$

where $\omega = v_x - u_y$ is the vorticity.

Among the periodic waves there are the steady periodic waves: given a wave speed $c > 0$, the dependence of the free surface, of the pressure, and of the velocity field has the form $x - ct$. For such waves, Euler's equation (A 2) implies that $\psi - cy$ and ω are functionally dependent since their gradients are parallel. For simplicity, we assume that there exists a C^1 function γ, called the vorticity function, such that

$$\omega = \gamma(\psi - cy) \tag{A 8}$$

throughout the fluid. This is always the case if, for instance, the steady wave satisfies the condition

$$u < c \tag{A 9}$$

throughout the fluid domain (Constantin & Strauss 2004). Note that $\psi - cy$ and ω have the same level sets. In the moving frame, the domain Ω is independent of t. Under assumption (A 9), the governing equations can be reformulated in the moving frame where x replaces $x - ct$, as (6).

The theoretical understanding of water waves started with the work of Airy, Stokes and their contemporaries in the nineteenth century. While linearization about the rest state provided the first insights into the dynamics of water waves and led to the development of the linear theory, it was observed that actual water wave characterstics deviate significantly from the linear theory predictions. This motivated an extensive study of the nonlinear governing equations. Remarkable success was achieved within the irrotational framework ($\omega \equiv 0$) when it was observed that equation (A 7) establishes a link to harmonic functions and opens up the possibility of using complex function theory to study the water-wave problem. In the irrotational case, an outstanding achievement was the understanding of large-amplitude irrotational steady waves through a programme initiated by Stokes (1847) (see the review by Toland 1996).

The failure of these powerful methods for non-constant vorticity explains why the present theoretical understanding of water waves is almost exclusively restricted to irrotational flows. The assumption of irrotational flow is suitable for waves advancing into still water, but flows with non-constant vorticity are commonly seen in nature, for instance in the case of waves on running water (wave–current interactions) Johnson (1997). Concerning steady water waves with vorticity, a remarkable early paper (Gerstner 1809; Constantin 2001) gave the first explicit solution to the governing equations for infinite depth with a particular non-constant vorticity. For a general vorticity, some investigations (Dubreil-Jacotin 1934; Goyon 1958; Zeidler 1973) showed the existence of such waves that are close to a flat surface, but the existence of waves that are not nearly flat was only recently established (Constantin & Strauss 2002, 2004).

The invariants of a time-dependent flow are well-known and elementary (see for instance Longuet-Higgins 1983). Since Longuet-Higgins did not cite the invariant $\mathscr{F}$, we now complete his presentation with a quick derivation of it. Introducing the material derivative of the scalar function $f(t, x, y)$ by $\mathrm{D}f/\mathrm{D}t = f_t + uf_x + vf_y$, we easily see that

$$\frac{\mathrm{D}\omega}{\mathrm{D}t} = 0, \tag{A 10}$$

so that the vorticity of each particle is preserved. Then,

$$\frac{\mathrm{d}}{\mathrm{d}t}\int\int_{\Omega(t)} F(\omega)\,\mathrm{d}y\,\mathrm{d}x = \int\int_{\Omega(t)} \partial_t\{F(\omega)\}\,\mathrm{d}y\,\mathrm{d}x + \int_{S(t)} F(\omega)\,\eta_t\,\mathrm{d}x$$

$$= \int\int_{\Omega(t)} \left(\frac{\mathrm{D}}{\mathrm{D}t}\{F(\omega)\} - u\{F(\omega)\}_x - v\{F(\omega)\}_y\right)\mathrm{d}y\,\mathrm{d}x + \int_{S(t)} F(\omega)\,\eta_t\,\mathrm{d}x.$$

Notice that if $h(x, y)$ and $k(x, y)$ are periodic in the x-variable, then

$$\int\int_{\Omega(t)} h_x k\,\mathrm{d}y\,\mathrm{d}x = -\int\int_{\Omega(t)} hk_x\,\mathrm{d}y\,\mathrm{d}x - \int_{S(t)} \eta_x hk\,\mathrm{d}x.$$

This can easily be checked by applying Green's theorem to hk. Therefore, since (A10) yields $\mathrm{D}F(\omega)/\mathrm{D}t = 0$, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\int\int_{\Omega(t)} F(\omega)\,\mathrm{d}y\,\mathrm{d}x = \int\int_{\Omega(t)} (u_x + v_y)\,F(\omega)\,\mathrm{d}y\,\mathrm{d}x + \int_{S(t)} (\eta_x u - v + \eta_t)\,F(\omega)\,\mathrm{d}x = 0$$

in view of (A1), (A4) and the periodicity assumption.

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